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# A finite-time heat engine with fixed-capacity heat transfers

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**Abstract.** We consider a finite-time heat engine in which the temperature $T(t)$ of the working fluid during heat exchanges with the hot and cold reservoirs is determined passively by the heat capacity of the fluid. The set of feasible operations of this engine, whose irreversibility arises solely from the temperature difference between the reservoir(s) and the working fluid, is described in terms of an inequality which sets limits on the rate $P$ of work output and the rate $D$ of entropy production associated with a cycle that takes place in time $\tau$. Those operations that lie on the boundary of this set are the ones that achieve a specified work output and entropy production in minimum time; this leads naturally to a notion of time efficiency for any operation within the set. The results obtained here extend our previous framework for Carnot-like processes to examine the time efficiency, as well as the power efficiency, of the corresponding Otto- and Brayton-cycle based engines. The present results and the earlier ones can be seen as two extremes of a continuum in which the external control on the temperature of the working fluid varies between (i) passively allowing the $T(t)$ corresponding to a constant heat capacity response and (ii) actively achieving any desired $T(t)$.

## 1. Introduction

In recent years considerable progress has been made in understanding the performance of finite-time heat engines and in elucidating the rules that govern in-principle limits to energy conversion in such devices. Since finite-time heat engines are necessarily irreversible, their efficiency is considerably reduced below the corresponding Carnot value $\eta_R$ and, in fact, comes much closer to the efficiency of the more realistic engines actually employed in industry. This has motivated researchers to pursue the subject of finite-time thermodynamics with some zeal, as a result of which a significant body of knowledge relevant to this field has accumulated to date [1–3].

Even though other sources of irreversibility cannot be ignored [4, 5], major attention in this area has been given to heat engines whose irreversibility arises solely from the transfer of heat between thermal reservoirs and the working fluid across boundaries sustaining *finite* temperature differences [6]. Assuming that the heat exchange is driven by a heat source or a heat sink at a constant temperature, and that the temperature of the working fluid is treated as a controllable variable, then no generality is lost by taking the heat exchange to be isothermal. This is due to the fact that the optimal control of the process demands that we maintain the temperature of the working fluid fixed while it is conducting business with one reservoir or the other. Such control is usually achieved by coupling the working fluid to a work reservoir. In the present investigation we consider the case when the working

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fluid is subject to some constraint that precludes its temperature from being controllable. For instance, during an Otto-like (or Brayton-like) cycle with an ideal gas, the volume (or the pressure) of the gas is held fixed during the heat exchange; it is then necessary either to vary the temperature of the reservoir or to consider the process of heat transfer as *non-isothermal*. Here we re-examine the case where the variation in the temperature of the working fluid during the heat exchange branch is well modelled by a constant heat capacity response. Note that this includes the Otto and Brayton cycles, provided the $C_V$ and the $C_p$ of the working fluid are constant. Similar problems have been treated before in the papers of Landsberg and Leff [7] and of Gordon and Huleihil [8]. There is an important difference between these two studies, however. While Gordon and Huleihil considered a finite-time, irreversible engine in which the temperature, $T_{\text{ext}}(t)$, of the external body (source or sink) is related to but differs from the temperature, $T(t)$, of the working fluid by a finite amount, Landsberg and Leff looked at a quasi-static, reversible engine in which $T_{\text{ext}}$, at all times, differs infinitesimally from $T$; of course, in each case both $T_{\text{ext}}$ and $T$ vary with time, making the processes of heat transfer non-isothermal.

In the light of these studies we decided to extend our own recent analysis of finite-time heat engines [9, 10] (hereafter referred to as I and II, respectively) to include the possibility of non-isothermal heat transfers. We preferred to stick to the notion of *fixed* temperatures for the reservoirs and permit only the temperature of the working fluid to be time-dependent; the temperature difference between the system and the reservoir(s) is then necessarily finite and the heat transfer processes inevitably irreversible. Maintaining the characteristic feature of our approach, namely the adoption of a reservoir-oriented viewpoint and the establishment of inequalities governing the outcome of finite-time processes, we analysed the problem of heat engines involving non-isothermal heat transfers in terms of quantities that are of direct practical concern. In section 2 we set up our basic inequality governing the entropy changes, $\sigma_1^0$ and $\sigma_2^0$, of the two reservoirs in terms of the time, $\tau$, of the cycle of processes involved. This serves to delimit the physically feasible cycles in terms of their net effects. In section 3 we render this inequality into a more practical one in terms of the *power* $P$ and the *degradation* $D$ associated with the cycle, and examine some of its more obvious consequences. In section 4, $P$ and $D$ are expressed in terms of a natural set of parameters—$\alpha$, related to the allocation of time between the two heat exchange branches, and $\beta$, a parameter introduced in section 3 and related to power-efficiency. Section 5 explores the connection between improved performance and a higher degree of adiabatic control of the working fluid. Here the improvement is reflected in the fact that the same net effects are achieved in a shorter time. This sets the stage for a discussion of time-efficiency in section 6. Section 7 looks at certain limiting cases and makes some connections to the previous work of other authors.

## 2. The basic inequality

We consider a cycle of processes to which a given heat engine is subjected and examine the net environmental changes brought about by the cycle. We first focus our attention on the process of heat transfer between the hot reservoir at temperature $T_1^0$ (assumed *fixed*) and the working fluid, which will be regarded as our thermodynamic system, at temperature $T_1$ (assumed *variable*); see figure 1. Assuming the heat-transfer law to be a *linear* one, the rate of flow of heat into the system is given by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \kappa(T_1^0 - T_1) \tag{1}$$

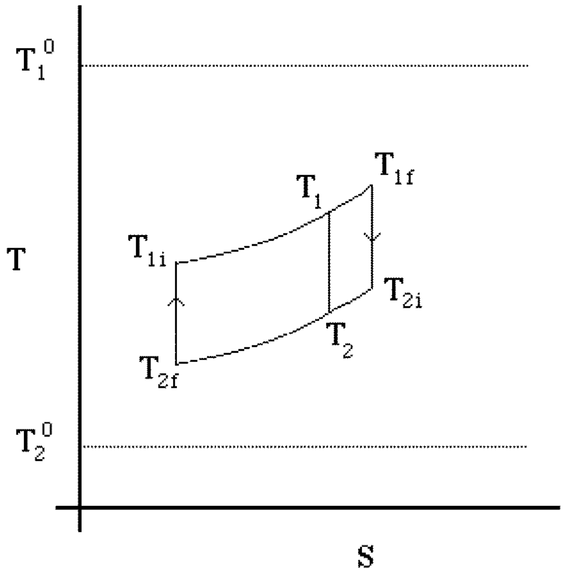


**Figure 1.** The cycle undergone by the working fluid, showing temperatures at various stages of the cycle. Note that, for all $S$, the ratio $T_2(S)/T_1(S)$ is independent of $S$.

where $\kappa$ is the thermal conductance of the surface of contact between the reservoir and the system. The resulting rate of change of the system temperature $T_1$ is given by

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \frac{\kappa}{C}(T_1^0 - T_1) \tag{2}$$

where $C$ is the heat capacity of the system. Denoting the initial value of $T_1$ by $T_{1i}$ and the duration of the process by $\tau_1$, the final value of $T_1$ on this branch of the cycle turns out to be

$$T_{1f} = T_1^0 - (T_1^0 - T_{1i})\mathrm{e}^{-\kappa\tau_1/C}. \tag{3}$$

The accompanying changes in the entropy of the system and of the reservoir, assuming endoreversibility, are given by

$$\sigma_1 = C\ln(T_{1f}/T_{1i}) \tag{4}$$

and

$$\sigma_1^0 = -C\frac{T_{1f} - T_{1i}}{T_1^0} \tag{5}$$

respectively. Eliminating the temperature variables among these equations, we obtain

$$\frac{C}{\sigma_1^0} = (1 - \mathrm{e}^{\sigma_1/C})^{-1} - (1 - \mathrm{e}^{-\kappa\tau_1/C})^{-1}. \tag{6}$$

Similar considerations may be applied to the process of heat transfer between the cold reservoir at temperature $T_2^0$ and the system at temperature $T_2(t)$, with the results

$$T_{2f} = T_2^0 + (T_{2i} - T_2^0)\mathrm{e}^{-\kappa\tau_2/C} \tag{7}$$

$$\sigma_2 = -C\ln(T_{2i}/T_{2f}) \tag{8}$$

and

$$\sigma_2^0 = C\,\frac{T_{2i} - T_{2f}}{T_2^0} \tag{9}$$

where $\sigma_2$ and $\sigma_2^0$ are the entropy changes of the system and of the cold reservoir on the lower branch of the cycle while $\tau_2$ is the duration of the corresponding process. Eliminating $T$s from equations (7)–(9), we get

$$\frac{C}{\sigma_2^0} = (1 - \mathrm{e}^{\sigma_2/C})^{-1} - (1 - \mathrm{e}^{-\kappa\tau_2/C})^{-1}. \tag{10}$$

Finally, combining equations (6) and (10), and remembering that the net change, $\sigma_1 + \sigma_2$, in the entropy of the system is zero, we obtain

$$\frac{C}{\sigma_1^0} + \frac{C}{\sigma_2^0} = 1 - (1 - \mathrm{e}^{-\kappa\tau_1/C})^{-1} - (1 - \mathrm{e}^{-\kappa\tau_2/C})^{-1} = -\frac{\sinh(\kappa\tau/2C)}{2\sinh(\kappa\tau_1/2C)\sinh(\kappa\tau_2/2C)} \tag{11}$$

where $\tau = \tau_1 + \tau_2$. Assuming the adiabatic processes linking the upper and lower branches of the cycle to be instantaneous, $\tau$ becomes the total time of the cycle. Agrawal *et al* have made a study in which this assumption is relaxed [11].

It is quite straightforward to see that, for a given $\tau$, the right-hand side of (11) is maximum when $\tau_1 = \tau_2 = \tau/2$. This leads to our basic inequality

$$\frac{C}{\sigma_1^0} + \frac{C}{\sigma_2^0} \leqslant -\coth\left(\frac{\kappa\tau}{4C}\right). \tag{12}$$

For $\tau \ll C/\kappa$, expression (12) becomes independent of $C$, namely

$$\frac{1}{\sigma_1^0} + \frac{1}{\sigma_2^0} \leqslant -\frac{4}{\kappa\tau} \tag{13}$$

which agrees with our earlier result

$$\sigma_1^0 + \sigma_2^0 \geqslant -4\sigma_1^0\sigma_2^0/\kappa\tau \tag{14}$$

pertaining to a cycle with *isothermal* heat transfers; see equation (2) of II. This agreement is not surprising because, for $\tau \ll C/\kappa$, the variation in the temperature $T_1$ of the system along the upper branch and the variation in $T_2$ along the lower branch of the cycle are quite negligible—making heat transfer processes essentially isothermal.

At this point we are tempted to introduce an *effective* thermal conductance $\kappa^*$, defined by

$$\kappa^* = (4C/\tau)\tanh(\kappa\tau/4C). \tag{15}$$

This enables us to write (12) in a form similar to (13), namely

$$\frac{1}{\sigma_1^0} + \frac{1}{\sigma_2^0} \leqslant -\frac{4}{\kappa^*\tau}. \tag{16}$$

Consequently, many of the results reported in I and II can be adapted to the present problem by simply replacing $\kappa$ by $\kappa^*$. In figure 2 we have plotted $\kappa^*$ as a function of $\tau$. We note that, for $\tau \ll C/\kappa$, $\kappa^* \cong \kappa$; however, as $\tau$ increases, $\kappa^*$ decreases monotonically and, as $\tau$ becomes large in comparison with $C/\kappa$, $\kappa^*$ vanishes as $\tau^{-1}$. Thus, the effective conductance is highest in the limit $\tau \to 0$. An interpretation of this result will be given in section 5.

An interesting consequence of the fact that our heat transfer processes are non-isothermal is seen when our inequality is written in a form similar to (14); we then have

$$\sigma_1^0 + \sigma_2^0 \geqslant -4\sigma_1^0\sigma_2^0/\kappa^*\tau \tag{17}$$


**Figure 2.** The effective thermal conductance $\kappa^*$ as a function of the cycle time $\tau$. Note that as $\tau$ tends to zero, $\kappa^*$ approaches $\kappa$.

which sets a positive lower bound on the total entropy production of the cycle. In the limit $\tau \to \infty$, this becomes

$$\sigma_1^0 + \sigma_2^0 \geqslant -\sigma_1^0\sigma_2^0/C \tag{18}$$

which differs rather significantly from the corresponding result, $\sigma_1^0 + \sigma_2^0 \geqslant 0$, pertaining, in the same limiting case, to a cycle with isothermal heat transfers. Clearly, the bound set by (18) arises from the fact that, in the case of non-isothermal heat transfers, even for $\tau \to \infty$, the temperature difference between the system and the reservoir(s) remains finite for much of the duration of the cycle; the processes involved are, therefore, irreversible and the quantity $(\sigma_1^0 + \sigma_2^0)$ strictly positive.

## 3. Thermodynamic limitations of the finite-time cycle

We shall now examine the implications of the inequality (17) in terms of the *power* $P$ (which denotes the average rate at which work is performed in the cycle) and the *degradation* $D$ (which denotes the average rate at which the entropy of the universe increases during the cycle):

$$P = \frac{W}{\tau} = \frac{Q_1 - Q_2}{\tau} = \frac{-T_1^0\sigma_1^0 - T_2^0\sigma_2^0}{\tau} \tag{19}$$

and

$$D = \frac{\sigma_1^0 + \sigma_2^0}{\tau} \tag{20}$$

which lead to the inverse relationships

$$\sigma_1^0 = -\tau\frac{P + T_2^0 D}{T_1^0 - T_2^0} \tag{21$a$}$$

$$\sigma_2^0 = \tau\frac{P + T_1^0 D}{T_1^0 - T_2^0}. \tag{21$b$}$$

Substituting (21) into (17), we get

$$(P + T_1^0 D)(P + T_2^0 D) - \tfrac{1}{4}\kappa^*(T_1^0 - T_2^0)^2 D \leqslant 0. \tag{22}$$

With $T_1^0$, $T_2^0$ and $\kappa^*$ given, the inequality (22) sets definitive limits on the values that the quantities $P$ and $D$ can have in any cycle of operations conducted in time $\tau$. Employing the $(P, D)$ plane as the forum for distinguishing between the outcome of one cycle and that of another, see figure 3, we observe that the limitations imposed by (22) confine the feasible values of $P$ and $D$ to the space *within* and *on* the hyperbola defined by the equality in (22). Thus, for any specified value of $P$,

$$D_1(P) \leqslant D \leqslant D_2(P) \tag{23}$$

where $D_1$ and $D_2$ are the roots of the quadratic equation representing the hyperbola. The power generated is *maximum* when $D_1(P) = D_2(P)$; this happens in a cycle depicted by the point A, with

$$P_{\max} = \tfrac{1}{4}\kappa^*\left(\sqrt{T_1^0} - \sqrt{T_2^0}\right)^2 \tag{24a}$$

$$D(P_{\max}) = \frac{P_{\max}}{\sqrt{T_1^0 T_2^0}} = \tfrac{1}{4}\kappa^*\left(\sqrt[4]{\frac{T_1^0}{T_2^0}} - \sqrt[4]{\frac{T_2^0}{T_1^0}}\right)^2. \tag{24b}$$

To determine the efficiency of this cycle as well as of others, we proceed as follows.

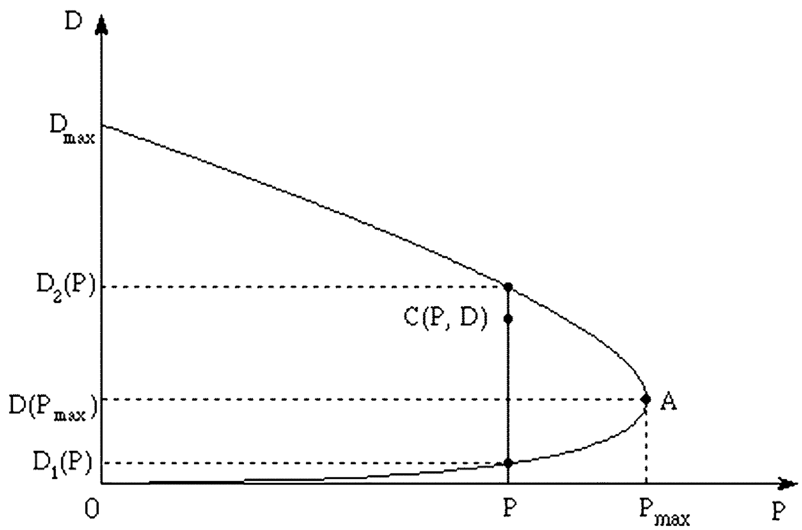


**Figure 3.** Representation of a cyclic process $C$ in the $(P, D)$ plane. The point $A$ pertains to the production of maximum power.

Quite generally, the power efficiency $\eta$ of a cycle is given by

$$\eta = \frac{P\tau}{-T_1^0\sigma_1^0} = \frac{T_1^0 - T_2^0}{T_1^0}\frac{P}{P + T_2^0 D} = \eta_R \frac{1}{1 + T_2^0 m} \tag{25}$$

where $\eta_R(= 1 - T_2^0/T_1^0)$ is the efficiency of the corresponding Carnot cycle, while $m(= D/P)$ is the slope of the straight line joining the origin O with the representative point $(P, D)$ of the cycle under study; see figure 4. With $T_1^0$ and $T_2^0$ given, the power efficiency of the cycle is determined solely by the slope of this line and, in consequence, the 'lines of constant $\eta$ in this picture are the radial lines emanating from the origin'. The efficiency of the cycle with *maximum* power production is determined by the slope of the line OA which, by equation (24*b*), is $1/\sqrt{T_1^0 T_2^0}$; consequently,

$$\eta(P_{\max}) = \eta_R \frac{1}{1+\sqrt{T_2^0/T_1^0}} = 1 - \sqrt{\frac{T_2^0}{T_1^0}} \tag{26}$$

which turns out to be the same as the standard Novikov–Curzon–Ahlborn result [12, 13].

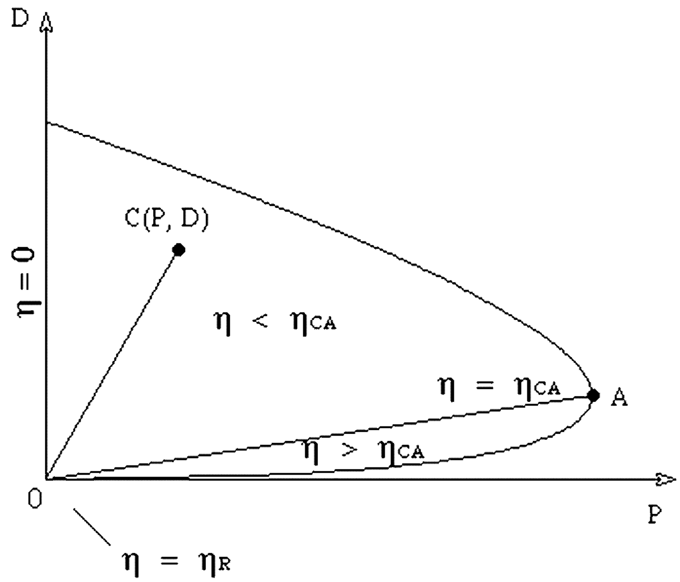


**Figure 4.** The power efficiency $\eta$ of a cyclic process $C$ is determined by the slope of the line OC. The line OA, along which $\eta$ has the Curzon–Ahlborn value $\eta_{CA}$, divides the allowed space into two parts—the upper one with $\eta < \eta_{CA}$ and the lower one with $\eta > \eta_{CA}$.

For a broader understanding of the situation, we may relate the power efficiency $\eta$ with the temperatures, $T_1$ and $T_2$, of the system itself as it goes through the cycle. In this context, we note that the system temperatures at the four corners of the cycle, see figure 1, are mutually related by the fact that the entropy excursion, $\sigma_1$, on the upper branch of the cycle is the same as the one, $|\sigma_2|$, on the lower branch. Equations (4) and (8) then give

$$\frac{T_{1f}}{T_{1i}} = \frac{T_{2i}}{T_{2f}} \tag{27}$$

so that

$$\frac{T_{2f}}{T_{1i}} = \frac{T_{2i}}{T_{1f}} = \beta \tag{28}$$

say. A little reflection shows that the temperature $T_2$ anywhere on the lower branch of the cycle bears the same ratio, $\beta$, to the temperature $T_1$ directly above it. The quantity $\beta$ is, therefore, a well defined parameter of the problem—despite the fact that both $T_1$ and $T_2$ vary with time. Now, since the amounts of heat, $Q_1$ and $Q_2$, transferred during the cycle are given by

$$Q_1 = -T_1^0\sigma_1^0 = CT_{1i}\left(\frac{T_{1f}}{T_{1i}} - 1\right) \tag{29}$$

and

$$Q_2 = T_2^0 \sigma_2^0 = C T_{2f} \left( \frac{T_{2i}}{T_{2f}} - 1 \right) \tag{30}$$

respectively, see equations (5) and (9), the power efficiency of the cycle is also given by

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_{2f}}{T_{1i}} = 1 - \beta \tag{31}$$

(see equations (27) and (28) as well). Equating (25) and (31), we obtain a direct correspondence between the quantities $\beta$ and $m$, namely

$$\beta = \frac{\beta^0 + T_2^0 m}{1 + T_2^0 m} \tag{32a}$$

$$m = \frac{\beta - \beta^0}{1 - \beta} \frac{1}{T_2^0} \tag{32b}$$

where $\beta^0 = T_2^0 / T_1^0$. It follows that the radial lines in figure 4 are also lines of constant $\beta$.

## 4. Alternative representation of the cyclic process

We shall now express our principal quantities $P$ and $D$ in terms of the parameter $\beta$ introduced in section 3 and the times $\tau_1$ and $\tau_2$ spent by the system in contact with the reservoirs 1 and 2, respectively. For this we substitute expressions (21) for $\sigma_1^0$ and $\sigma_2^0$ into (11), to obtain

$$P = \frac{C}{\tau} \frac{(T_1^0 - T_2^0)^2 m}{(1 + T_1^0 m)(1 + T_2^0 m)} \frac{2 \sinh(\kappa \tau_1 / 2C) \sinh(\kappa \tau_2 / 2C)}{\sinh(\kappa \tau / 2C)}. \tag{33}$$

Next, we use relation (32*b*) to eliminate $m$ in favour of $\beta$, to get

$$P = \frac{C}{\tau} T_1^0 \frac{(\beta - \beta^0)(1 - \beta)}{\beta} \frac{2 \sinh(\kappa \tau_1 / 2C) \sinh(\kappa \tau_2 / 2C)}{\sinh(\kappa \tau / 2C)}. \tag{34}$$

Finally we utilize expression (15) for the effective conductance $\kappa^*$, to write

$$P = \frac{1}{4} \kappa^* T_1^0 \alpha \frac{(\beta - \beta^0)(1 - \beta)}{\beta} \tag{35}$$

where

$$\alpha = \frac{\sinh(\kappa \tau_1 / 2C) \sinh(\kappa \tau_2 / 2C)}{\sinh^2(\kappa \tau / 4C)} \leqslant 1 \tag{36}$$

equality holding when $\tau_1 = \tau_2 = \tau/2$. It follows that

$$D = mP = \frac{1}{4} \kappa^* \alpha \frac{(\beta - \beta^0)^2}{\beta \beta^0}. \tag{37}$$

Clearly, $\alpha$ and $\beta$ constitute a natural set of parameters for describing the cycle under study. We shall now make some observations on the foregoing results.

First of all, the dependence of $P$ on $\beta$ is such that, for any given $\alpha$, $P$ is maximum when $\beta = \sqrt{\beta^0}$, giving

$$\tilde{P} = \tfrac{1}{4} \kappa^* \alpha \left( \sqrt{T_1^0} - \sqrt{T_2^0} \right)^2 \tag{38a}$$

$$D(\tilde{P}) = \tfrac{1}{4} \kappa^* \alpha \left( \sqrt[4]{T_1^0 / T_2^0} - \sqrt[4]{T_2^0 / T_1^0} \right)^2 \tag{38b}$$

(cf equations (24$a$, $b$), which correspond to the optimal case $\alpha = 1$). Next, for any given $\beta$, both $P$ and $D$ are directly proportional to $\alpha$. Clearly, this dependence stems from the 'manner in which the total time $\tau$ is split into the time intervals $\tau_1$ and $\tau_2$ spent on the upper and lower branches of the cycle, respectively'. The more even the split, the larger is $\alpha$; in the limit when $\tau_1$ and $\tau_2$ become equal, $\alpha$ assumes its largest possible value of 1 and, with it, $P$ and $D$ assume their largest possible values consistent with the given value of $\beta$. Of course, if $\alpha = 1$ and $\beta = \sqrt{\beta^0}$, we recover the truly optimal value of $P$, namely the one given by (24$a$).

Our result for the maximum power turns out to be larger than the one obtained by Gordon and Huleihil [8]. Unfortunately, their argument contains a subtle flaw. They make inappropriate use of an equation that relates a temperature profile $T(t)$ of the working fluid with an associated profile $T^0(t)$ for the temperature of the heat source or sink. That equation correctly provides the optimal $T(t)$ for a specified $T^0(t)$. They, however, apply the equation in reverse by specifying the $T(t)$ appropriate for an isochore and then claiming that the equation provides the optimal $T^0(t)$ for a constant-volume process. To see the flaw in their argument in a somewhat simpler context, consider a function $f(x, y)$ of two variables. The maximum of $f$ with respect to $y$ for a specified $x = x_0$ may yield an optimal value $y^*$ of $y$, but the maximum of $f$ with respect to $x$ for given $y = y^*$ may not be at $x_0$. Try, for example, $f(x, y) = x^2 + xy + y^2$, with $x_0 = 2$. The roles of $T^0(t)$ and $T(t)$ in the Gordon–Huleihil argument are played here by the variables $x$ and $y$, respectively.

## 5. Control of the working fluid and optimal performance

As can be seen from the factor $\kappa^*$ in equation (35) and figure 2, the power of these engines is a decreasing function of the cycle time $\tau$. Indeed, the maximum power occurs only in the limit as the cycle time $\tau$ tends to zero! In the discussion below, we will see that this result can be reinterpreted to show that, for machines operating in a given cycle time, it is the increased control of the working fluid that results in an improved performance.

In the processes considered here the state of the working fluid is continually changing but that change is *actively* controllable (in terms of the operating parameters) only during the two (instantaneous) adiabatic branches, when the temperature of the working fluid is effectively initialized in preparation for the following heat exchange; during the heat exchange itself, the state of the working fluid is *passively* determined by its relaxation dynamics with respect to the heat source or sink.

In such engines, the working fluid is under minimal control. Suppose additional control is made possible by allowing the temperature of the working fluid to be reset adiabatically at discrete times during the heat exchange. For instance, let the working fluid, in its encounters with each of the reservoirs, experience $n$ passive episodes of equal duration, $\tau/2n$, punctuated by $n - 1$ discrete instants when its temperature is restored adiabatically to the initial value, $T_{1i}$ (or $T_{2i}$), that marked the onset of its interface with the current hot reservoir 1 (or cold reservoir 2); as before, the encounter with each reservoir is terminated with an adiabatic transition to the other, resulting in $2n$ adiabats altogether (see figure 5). The sequence of passive heat-exchange episodes might be denoted schematically by

$$H_1 H_2 \ldots H_n C_1 C_2 \ldots C_n \tag{39}$$

where $H$ and $C$ represent contact with reservoirs 1 and 2, respectively.

Since the duration, $\tau/2n$, of each episode is specified, the dynamic temperature profile of the working fluid during the sequence is completely determined by the choice of the two parameters $T_{1i}$ and $T_{2i}$. However, since (39) is to represent one cycle of the operation of a

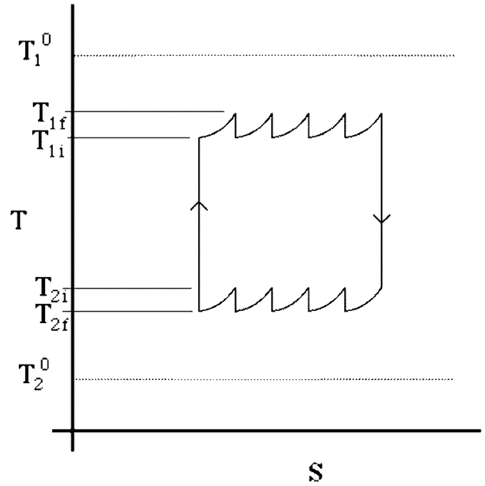


**Figure 5.** A representation of the cyclic process (39) containing 10 adiabats. This is equivalent to the process (40), which consists of 5 cycles.

heat engine, the entropy, as well as the temperature, of the working fluid must be restored at the end of the sequence, i.e. the temperatures $T_{1i}$ and $T_{2i}$ cannot be chosen independently. It is enough that they be chosen to satisfy the condition $\sigma_1 + \sigma_2 = 0$, where $\sigma_1$, respectively $\sigma_2$, represents the entropy change of the working fluid during any of the (identical) $H$, respectively $C$, episodes.

Now since the net effects of the cycle, i.e. the work and entropy production, depend only on the heat-exchange episodes, this sequence can be reordered in time, with the same net effects, to produce $n$ successive cycles, $H_k C_k$, of the minimal-control type that we have studied in this paper, namely

$$H_1 C_1 H_2 C_2 \ldots H_n C_n. \tag{40}$$

Each pair $H_k C_k$ is itself a cycle, as a result of the condition $\sigma_1 + \sigma_2 = 0$ imposed above. The important point is that the power produced by the cycle (39) is the same as that produced by the sequence of cycles (40) and, hence, the same as that produced by each constituent cycle $H_k C_k$.

The shift in viewpoint from (39) to (40) enables us to understand how the added control in (39) affects its optimal performance as an engine with cycle time $\tau$; the additional adiabats in (39) reduce its 'effective' cycle time to that of $H_k C_k$. For example, if each $H_k C_k$ in (40) produces maximal power for its cycle time of $\tau/n$, then the power produced by (39) is given by, see (24$a$):

$$P_{\max}(\tau, n) = \tfrac{1}{4}\kappa^*(\tau/n)\left(\sqrt{T_1^0} - \sqrt{T_2^0}\right)^2. \tag{41}$$

Now in the limit as $n \to \infty$ we see from figure 2 that $\kappa^*(\tau/n) \to \kappa$ and (41) approaches the well known result of Novikov–Curzon–Ahlborn [12, 13], which gives the maximum power achievable when the temperature of the working fluid is fully controllable. The freedom to insert an infinite number of adiabats brings with it the choice to come arbitrarily close to an isothermal operation. The present way of achieving this is along the lines adopted in [2], where the distribution of contacts is the quantity optimized.

## 6. Time efficiency

Consider a cyclic process operating for a time $t$ of, perhaps, many cycles. In paper II we introduced the concept of *time-efficiency* of such a process as a measure of how efficient the process is when compared to a similar one that yields the same work output $W(= Pt)$ and the same entropy production $\Delta S^u(= Dt)$ in the *shortest* possible time $t^*$. The time-efficiency of the process in question is then given by $\theta = t^*/t$.

It is straightforward to see that if the cyclic process pertaining to $t^*$ is denoted by the point $(P^*, D^*)$ and the one under study by $(P, D)$, then the ratios $t^*/t$, $P/P^*$ and $D/D^*$ are all equal. The value of $P$ is given by equation (35); the value of $P^*$ is obtained from (35) by setting $\alpha$ and $\kappa^*$ equal to their maximal values—namely 1 and $\kappa$, respectively. Since both processes produce the same net effects in their respective times, $P/D$ and $P^*/D^*$ are equal; hence the parameter $\beta$ is the same for both processes. It follows that the time-efficiency of a (two-adiabat) cyclic process of the type that we have discussed, with branch times $\tau_1$ and $\tau_2$ and total cycle time $\tau$, is given by

$$\theta(\tau_1, \tau_2) = P/P^* = \alpha(\tau_1, \tau_2)\gamma(\tau) \tag{42}$$

where we have found it useful to introduce the dimensionless parameter $\gamma(\tau) = \kappa^*(\tau)/\kappa$.

The factor $\alpha(\tau_1, \tau_2)$, see equation (36), measures the efficiency of the time allocation between the two heat-exchange branches; its maximum value is 1 when $\tau_1 = \tau_2 = \tau/2$, independently of the cycle time $\tau$. The second factor $\gamma(\tau)$, see equation (15), can be interpreted as the maximum time-efficiency realizable by (two-adiabat) processes of cycle time $\tau$. More generally, in view of the discussion in the previous section, the maximum time efficiency for a cycle with $2n$ adiabats is given by

$$\gamma(\tau/n) = \frac{4nC}{\kappa\tau}\tanh\left(\frac{\kappa\tau}{4nC}\right). \tag{43}$$

## 7. Concluding remarks

The fact that the heat-transfer processes in our model are *non-isothermal* affects our final results in two distinct ways—first by introducing an effective conductance $\kappa^*$, as in equation (15), and second by modifying the form of the function $\alpha(\tau_1, \tau_2)$, as in equation (36). Whereas $\alpha$ can be made as large as 1 (simply by choosing $\tau_1 = \tau_2$) *regardless* of the total time $\tau$, $\kappa^*$ depends rather crucially on $\tau$; see figure 2. For $\tau \ll C/\kappa$, we recover results pertaining to the case of *isothermal* heat transfers, for then

$$\kappa^* \to \kappa \tag{44a}$$

$$\alpha \to \frac{4\tau_1\tau_2}{\tau^2}. \tag{44b}$$

This yields the maximum possible power insofar as conductivity is concerned, though the total work output in this case diminishes proportionately with $\tau$; this observation agrees with a similar one made by Feldmann *et al* [14] in connection with a finite-time heat engine in which the working medium consists of non-interacting two-level systems. On the other hand, for $\tau \gg C/\kappa$, $\kappa^* \to 4C/\tau$, with the result that now the quantities $P$ and $D$ tend to vanish while the total work output $W$ and the total entropy production $\Delta S^u$ tend to finite values

$$W = CT_1^0\alpha\frac{(\beta - \beta^0)(1 - \beta)}{\beta} \tag{45a}$$

$$\Delta S^u = C\alpha \frac{(\beta - \beta^0)^2}{\beta\beta^0}. \qquad (45b)$$

The fact that our working fluid has a *finite* heat capacity expresses its influence most vividly in this limit.

Before closing this section we would like to emphasize that, for a given set of values of the parameters $\alpha$ and $\beta$, the system temperatures $T_{1i}$, $T_{1f}$, $T_{2i}$ and $T_{2f}$ are all well defined—except for a *duplicity* arising from the allocation of the time intervals $\tau_1$ and $\tau_2$ to one heat-transfer branch or the other. With $\tau_1$ specified, the temperatures $T_{1i}$ and $T_{1f}$ are related by equation (3) and with $\tau_2$ specified, the temperatures $T_{2i}$ and $T_{2f}$ are related by equation (7). At the same time, we have two relations, in equations (28), that bind these quantities through $\beta$. The net result is that, with no choice left to us,

$$T_{1i} = \frac{T_1^0(1 - e^{-\kappa\tau_1/C})e^{-\kappa\tau_2/C} + (T_2^0/\beta)(1 - e^{-\kappa\tau_2/C})}{1 - e^{-\kappa\tau/C}} \qquad (46a)$$

$$T_{1f} = \frac{T_1^0(1 - e^{-\kappa\tau_1/C}) + (T_2^0/\beta)(1 - e^{-\kappa\tau_2/C})e^{-\kappa\tau_1/C}}{1 - e^{-\kappa\tau/C}} \qquad (46b)$$

$$T_{2i} = \beta T_{1f} \qquad (46c)$$

$$T_{2f} = \beta T_{1i}. \qquad (46d)$$

For $\tau_1$, $\tau_2 \ll C/\kappa$, we recover results pertaining to isothermal heat transfers, namely

$$T_{1i} \cong T_{1f} \cong T_1^0 \frac{\tau_1}{\tau} + \frac{T_2^0}{\beta}\frac{\tau_2}{\tau} \qquad (= T_1,\ \text{say}) \qquad (47a)$$

$$T_{2i} \cong T_{2f} \cong \beta T_1 \qquad (= T_2,\ \text{say}). \qquad (47b)$$

On the other hand, for $\tau_1$, $\tau_2 \gg C/\kappa$, we have instead

$$T_{1i} \cong T_2^0/\beta \qquad T_{1f} \cong T_1^0 \qquad (48a)$$

$$T_{2i} \cong \beta T_1^0 \qquad T_{2f} \cong T_2^0. \qquad (48b)$$

The total work output is then given by

$$W = Q_1 - Q_2 = C\{(T_{1f} - T_{1i}) - (T_{2i} - T_{2f})\} \cong C(T_1^0 + T_2^0 - \beta T_1^0 - \beta^{-1}T_2^0) \qquad (49)$$

which agrees with the limit stated in (45*a*), since $\alpha$ in the present case is essentially equal to 1. Expression (49) is *maximum* when $\beta = \sqrt{T_2^0/T_1^0}$, making

$$T_{1i} \cong T_{2i} \cong \sqrt{T_1^0 T_2^0} \cong \sqrt{T_{1f}T_{2f}} \qquad (50)$$

with the corresponding

$$W \cong C\left(\sqrt{T_1^0} - \sqrt{T_2^0}\right)^2 \qquad (51a)$$

$$Q_1 \cong C\left(T_1^0 - \sqrt{T_1^0 T_2^0}\right) \qquad (51b)$$

and $\eta$ the same as in (26). This case resembles most closely the one studied by Landsberg and Leff [7], though there is a vital difference between the two. While the cycle employed by Landsberg and Leff is a *reversible* one (taking an infinite time to accomplish and requiring, on each heat-transfer branch, a sequence of reservoirs with continually varying $T^0$), the cycle studied here is a finite-time *irreversible* one (requiring only two reservoirs at temperatures $T_1^0$ and $T_2^0$, to which the system temperatures $T_1$ and $T_2$ tend as $\tau$ becomes large). Consequently, while the net entropy production, $\Delta S^u$, in their case is zero, we have a finite entropy production, given by

$$\Delta S^u = -\frac{C(T_{1f} - T_{1i})}{T_1^0} + \frac{C(T_{2i} - T_{2f})}{T_2^0} \cong C\left(\sqrt[4]{\frac{T_1^0}{T_2^0}} - \sqrt[4]{\frac{T_2^0}{T_1^0}}\right)^2 \tag{52}$$

in agreement with the limit stated in (45*b*), with $\alpha = 1$ and $\beta = \sqrt{\beta^0}$.


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